JOURNAL OF DIFFERENTIAL EQUATIONS 9, 280–284 (1971)

Ultimate Boundedness Does Not Imply Almost Periodicity

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Received February 27, 1970

1. INTRODUCTION

The purpose of this note is to construct an example which answers a question of Seifert [1] in the negative. We construct an almost periodic differential equation, for which all solutions are uniformly-ultimately bounded, but none are almost periodic. Opial [2] has constructed an example of an almost periodic equation with all of its solutions bounded but none almost periodic. This example uses the construction of a singular flow on the torus which may be found either in Nemytskii and Stepanov [3] or Denjoy [4]. We modify Opial's example.

2. OPIAL'S EXAMPLE

We recall Opial's example and list the properties that we use. We start with a perfect nowhere dense set F and construct a flow on the torus whose cluster set is F. This is a so-called singular flow on the torus with rotation number ρ, which is irrational. We then can construct a continuous differential equation

$$x' = f(x, t) \tag{1}$$

with $f(x + 1, t) = f(x, t + 1) = f(x, t)$ which has no periodic solutions. Furthermore, if $y = x - \rho t$, then

$$y' = f(y + \rho t, t) - \rho = g(y, t) \tag{2}$$

is an equation with no almost periodic solutions. However, each solution y of this equation is bounded, in fact, each solution differs from its initial value

* This research was partially supported by The National Science Foundation under Grants GP 11623 at Iowa State University and CP 12261 at Case-Western Reserve University.

by at most 3, on the integers, see [5]. Furthermore, the equation is almost periodic in t uniformly for all y. We may pick f to be Lipschitz in x, see [4].

Two properties of Eq. (1) will be important. Let the complement of F in $(0, 1)$ be written as a disjoint union of open intervals $\bigcup_n (a_n, b_n)$. Then

$$\varphi(t; 0; a_n) - \varphi(t; 0; b_n) \to 0 \qquad \text{as} \quad t \to \pm\infty.$$

This is one of the properties that Opial uses. Next if $\eta \in F$ and η is not an a_n or a b_n, then $\varphi(t; 0; \eta)$ is a recurrent motion, see [6, p. 164]. That is, given $\epsilon > 0$, there exists a relatively dense set E_η such that for $t \in E_\eta$, we have $| \varphi(t; 0, \eta) - \eta | < \epsilon$. By uniform continuity of φ, E_η may be taken to be $\bigcup_{k=-\infty}^{\infty} (t_k - \delta, t_k + \delta)$ for some $\delta > 0$. The importance of this remark is that two such sets have a relatively dense intersection. This implies that, given $\epsilon > 0$, there exists a $\delta > 0$ such that $| \eta_1 - \eta_2 | < \delta$ and $\eta_1, \eta_2 \in F - \bigcup_n (\{a_n\} \cup \{b_n\})$ imply that $| \varphi(t; 0; \eta_1) - \varphi(t; 0; \eta_2) < \epsilon$ for t in a relatively dense set.

3. The Example

We now construct an example of an almost periodic equation with uniformly ultimately bounded solutions but no almost periodic solution. We take

$$y' = h(y, t) = \begin{cases} g(y, t) & | y | \leqslant 3, \\ g(y, t) + \dfrac{| y |}{y} C(y^2 - 9) & | y | > 3, \end{cases} \tag{3}$$

where C is a constant such that for $M = \sup_{y,t} | g(y, t) |$ we have $M + 7C < 0$.

Clearly $h(y, t)$ is continuous and almost periodic uniformly for all g. We now show that all solutions of (3) are uniformly ultimately bounded. Notice that $h(4, t) \leqslant M + 7C < 0$ and similarly $h(-4, t) > 0$ so that once a solution enters $| y | \leqslant 4$ it never leaves as t increases. We claim that 4 is a uniform ultimate bound. Let $B > 4$ be given, and $y(t)$ a solution such that $| y(t_0) | < B$, say $4 < y(t_0) < B$. Then as long as $y(t) \geqslant 4$ we have

$$y'(t) \leqslant M + C(y^2(t) - 9) = Cy^2(t) + (M - 9C)$$

so

$$y'(t) y^{-2}(t) \leqslant C + (M - 9C) y^{-2}(t) \leqslant C + (M - 9C)(16)^{-1} = D < 0.$$

Hence $(-y^{-1}(t))' \leqslant D$ for $t \geqslant t_0$ as long as $y(t) \geqslant 4$. Integrating and rearranging we get for such t, that $y(t) \leqslant [B^{-1} - D(t - t_0)]^{-1}$. Hence

$|y(t)| \leqslant 4$ for $t - t_0$ large uniformly for $0 < y(t_0) \leqslant B$. A similar argument holds for $-B \leqslant y(t_0) < 0$.

We now must show that no solution is almost periodic. Since y' is one sign in either of the regions $y > 4$ or $y < -4$, no almost periodic solution can be outside $|y| > 4$ at any point. No almost periodic solution can satisfy $|y| \leqslant 3$ since it would then be an almost periodic solution to (2). Finally we now show that no solution which is ever say in $3 < y < 4$ can be almost periodic.

Let z now denote any solution of (3) such that $3 < z(t) < 4$ for some t. Let y be any solution of (2) such that $y(t_0) = z(t_0) > -3$. We claim that $y(t) \leqslant z(t)$ for $t \leqslant t_0$ and $y(t) \leqslant z(t)$ for $t \geqslant t_0$. We first show this for $y(t_0) > 3$. Notice that $(y - z)'(t_1) = -C[y^2(t_0) - 9] > 0$ whenever $(y - z)(t_1) = 0$ and $y(t_1) > 3$. Hence $y(t) > z(t)$ for $t > t_0$ and t near t_0. Since they cannot cross in $|y| \leqslant 3$ they cannot cross on $t > t_0$ ($y(t_0) > 3$ implies $y(t) > -3$ for all t). A similar argument holds to the left. If $y(t_0) = 3$, then the result follows by continuity with respect to initial conditions since the above inequalities are strict. If $-3 < y(t_0) < 3$, then $y = z$ until $z(t) = 3$, then apply above. This implies in particular that no solution z of (3) enters both regions $y > 3$ and $y < -3$.

Let z be an almost periodic solution of (3). We show this leads to a contradiction. Since either $z(t) > 3$ or $z(t) < -3$ somewhere, we assume the former. Note that $z(0) > -3$. We pick out a solution $\bar{y}$ of (2) such that $\eta > \bar{y}(0)$ implies the solution of (2) starting at η crosses z or is always above z on $[0, \infty]$. In fact $\bar{y}(0) = \inf\{y(0) \mid y \text{ meets } z \text{ on } [0, \infty), y \text{ a solution of (2)}\}$. By continuity with respect to initial conditions $\bar{y}(t) \leqslant z(t)$ on $[0, \infty)$. We will arrive at a contradiction by showing that $\bar{y}(t) \leqslant z(t) - \epsilon$ for some $\epsilon > 0$ and that this cannot happen.

To show that an $\epsilon > 0$ exists we argue in the following way. Let $\sup_{t \geqslant 0} z(t) = 3 + b$, $b > 0$. Let $\epsilon_1(\alpha) = \sup\{|g(y_1, t) - g(y_2 t)| : t \in R, 3 + \alpha \leqslant y_1, y_2 \leqslant 3 + b\}$ for $0 \leqslant \alpha \leqslant b$. Notice that as $\alpha \to b$, $\epsilon_1(\alpha) \to 0$ so that there exists an $\alpha \in (0, b)$ such that $\epsilon_1(\alpha) + C[6\alpha + \alpha^2] = d < 0$. Fix such an α. Since z is almost periodic and $\sup z > 3 + \alpha$, there exists a relatively dense set

$$E = \bigcup_{k=1}^{\infty} (t_k - \delta, t_k + \delta)$$

such that $t \in E$ implies that

$$z(t) \geqslant 3 + \frac{\alpha + b}{2}.$$

Now if $0 < \eta < (b - \alpha)/2$, then $z(t) - \bar{y}(t) > \eta$ and $z(t) > 3 + \alpha$ imply that

$$(z - \bar{y})'(t) < \epsilon_1(\alpha) + C(z^2 - 9) < \epsilon_1(\alpha) + [6\alpha + \alpha^2]C = d < 0.$$

Consequently, if $z(t_k - \delta) - y(t_n - \delta) > \eta$ then this inequality holds on $(t_k - \delta, t_k + \delta)$. Hence $(x - \bar{y})(t_k + \delta) \leqslant \eta + 2d\delta$ which is negative if η is small and thus for η, small, $\bar{y}(t_k + \delta) > z(t_k + \delta)$ which can never happen. Thus $(z - \bar{y})(t_k - \delta)$ is bounded away from 0. This in fact implies that $(z - \bar{y})(t)$ is bounded away from 0 everywhere. For if not, let L be an inclusion interval for E. If $\eta > 0$ is given, suppose $(z - \bar{y})(t_0) < \eta\, e^{-ML}$ for some $t_0 > 0$, then consider the solutions $\bar{y}$ and y_1 of (2) with $y_1(t_0) = z(t_0)$. Here M is the Lipschitz constant for $g(t, y)$. Then

$$(y_1 - \bar{y})(t) \leqslant e^{k(t-t_0)}(y_1 - \bar{y})(t_0)$$

But since E has inclusion interval L, there exists a $t_k > t_0$ in E such that $(t_k - t_0) < L$. For this t_k we have $(y_1 - \bar{y})(t_k) < \eta$. But $\bar{y}(t_k) \leqslant z(t_k) \leqslant y_1(t_k)$. Since η was arbitrary we must have $\epsilon > 0$ such that $(z - y)(t) > \epsilon$ for all $t \geqslant 0$.

Now we show that this cannot happen. There are two cases. Either $\bar{y}(0)$ is in $F - \bigcup_n (\{a_n\} \cup \{b_n\})$ or not. If $\bar{y}(0)$ is not, then there exists numbers η_1 and η_2 such that $\eta_1 \leqslant \bar{y}(0) < \eta_2$, with η_1 and η_2 endpoints of an open interval in the complement of F. But if we let $\varphi(t, \eta)$ be the solution of (1) starting at η and $y(t, \eta)$ be the solution of (2) starting at η then

$$y(t, \eta_2) - y(t, \eta_1) = \varphi(t, \eta_2) - \varphi(t, \eta_1) \to 0$$

as $t \to \infty$ (see part 2). But $y(t, \eta_1) \leqslant \bar{y}(t) \leqslant z(t) \leqslant y(t, \eta_2)$. Consequently $z(t) - \bar{y}(t) \to 0$ as $t \to +\infty$.

If $\bar{y}(0) \in F$ and not an endpoint of the complement, then for given $\epsilon > 0$, let $\delta > 0$ be chosen so that $|\eta - \bar{y}(0)| < \delta$, and $\eta \in F$ and not an endpoint of the complement implies $|\varphi(t, \eta) - \varphi(t, \bar{y}(0))| < \epsilon$ for t in a relatively dense set. It is possible to find a δ by part 2, and an η since $\bar{y}(0)$ is a condensation point of F. As before this implies that $|y(t, \eta) - \bar{y}(t)| < \epsilon$ on a relatively dense set and hence $\bar{y}(t) \leqslant z(t) \leqslant y(t, \eta)$ implies that $z(t_0) - \bar{y}(t_0) < \epsilon$ for some $t_0 > 0$. This completes the contradiction and we conclude that z cannot be almost periodic.

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